SENSITIVITY AND BLOCK SENSITIVITY OF NESTED CANALYZING FUNCTION

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Abstract. Based on a recent characterization of nested canalyzing function (NCF), we obtain the formula of the sensitivity of any NCF. Hence we find that any sensitivity of NCF is between $\frac{n+1}{2}$ and n. Both lower and upper bounds are tight. We prove that the block sensitivity, hence the l-block sensitivity, is same to the sensitivity. It is well known that monotone function also has this property. We eventually find all the functions which are both monotone and nested canalyzing (MNCF). The cardinality of all the MNCF is also provided.

1. Introduction

Nested Canalyzing Functions (NCFs) were introduced recently in [9]. One important characteristic of (nested) canalyzing functions is that they exhibit a stabilizing effect on the dynamics of a system. That is, small perturbations of an initial state should not grow in time and must eventually end up in the same attractor of the initial state. The stability is typically measured using so-called Derrida plots which monitor the Hamming distance between a random initial state and its perturbed state as both evolve over time. If the Hamming distance decreases over time, the system is considered stable. The slope of the Derrida curve is used as a numerical measure of stability. Roughly speaking, the phase space of a stable system has few components and the limit cycle of each component is short.

It is shown in [7] that the class of nested canalyzing functions is identical to the class of so-called unate cascade Boolean functions, which has been studied extensively in engineering and computer science. It was shown in [3] that this class produces the binary decision diagrams with the shortest average path length. Thus, a more detailed mathematical study of this class of functions has applications to problems in engineering as well.

In [5], Cook et al. introduced the notion of sensitivity as a combinatorial measure for Boolean functions providing lower bounds on the time needed by CREW PRAM (concurrent read , but exclusive write (CREW) parallel random access machine (PRAM)). It was extended by Nisan [17] to block sensitivity. It is still open whether sensitivity and block sensitivity are polynomially related. Although the definition is straightforward, the sensitivity is understood only for a few classes function. For monotone function, The block sensitivity is same as its sensitivity.

Recently, in [13], a complete characterization for nested canalyzing function is obtained by obtaining its unique algebraic normal form. A new concept, called LAYER NUMBER, is introduced. Based on this, explicit formulas for number of all the nested canalyzing functions and the average sensitivity of any NCF are provided. Theoretically, It is showed why NCF is stable since the upper bound is a constant.

In this paper, we obtain the formula of the sensitivity of any NCF based on a characterization of NCF from [13]. We show the block sensitivity, like monotone function, is also same to its

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sensitivity. Finally, we characterize all the Boolean functions which are both nested canalyzing and monotone. We also give the number of such functions.

2. Preliminaries

In this section we introduce the definitions and notations. Let $\mathbb{F} = \mathbb{F}_2$ be the Galois field with 2 elements. If f is a n variable function from $\mathbb{F}^n$ to $\mathbb{F}$, it is well known [16] that f can be expressed as a polynomial, called the algebraic normal form(ANF):

$$f(x_1, x_2, \ldots, x_n) = \bigoplus_{0 \le k_i \le 1, i=1,\ldots,n} a_{k_1 k_2 \ldots k_n} {x_1}^{k_1} {x_2}^{k_2} \cdots {x_n}^{k_n}$$

where each coefficient $a_{k_1 k_2 \ldots k_n} \in \mathbb{F}$ is a constant.

Definition 2.1. *Let f be a Boolean function in n variables. Let σ be a permutation on $\{1, 2, \ldots, n\}$. The function f is nested canalyzing function (NCF) in the variable order $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$ with canalyzing input values $a_1, \ldots, a_n$ and canalyzed values $b_1, \ldots, b_n$, if it can be represented in the form*

$$f(x_1, \ldots, x_n) = \begin{cases} b_1 & x_{\sigma(1)} = a_1, \\ b_2 & x_{\sigma(1)} = \overline{a_1}, x_{\sigma(2)} = a_2, \\ b_3 & x_{\sigma(1)} = \overline{a_1}, x_{\sigma(2)} = \overline{a_2}, x_{\sigma(3)} = a_3, \\ \ldots\ldots \\ b_n & x_{\sigma(1)} = \overline{a_1}, x_{\sigma(2)} = \overline{a_2}, \ldots, x_{\sigma(n-1)} = \overline{a_{n-1}}, x_{\sigma(n)} = a_n, \\ \overline{b_n} & x_{\sigma(1)} = \overline{a_1}, x_{\sigma(2)} = \overline{a_2}, \ldots, x_{\sigma(n-1)} = \overline{a_{n-1}}, x_{\sigma(n)} = \overline{a_n}. \end{cases}$$

Where $\overline{a} = a \oplus 1$. The function f is nested canalyzing if f is nested canalyzing in the variable order $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$ for some permutation σ.

Theorem 2.1. [13] *Given $n \ge 2$, $f(x_1, x_2, \ldots, x_n)$ is nested canalyzing iff it can be uniquely written as*

$$f(x_1, x_2, \ldots, x_n) = M_1(M_2(\ldots(M_{r-1}(M_r \oplus 1) \oplus 1)\ldots) \oplus 1) \oplus b. \tag{2.1}$$

Where $M_i = \prod_{j=1}^{k_i}(x_{i_j} \oplus a_{i_j})$, $i = 1, \ldots, r$, $k_i \ge 1$ for $i = 1, \ldots, r-1$, $k_r \ge 2$, $k_1 + \ldots + k_r = n$, $a_{i_j} \in \mathbb{F}_2$, $\{i_j | j = 1, \ldots, k_i, i = 1, \ldots, r\} = \{1, \ldots, n\}$.

Because each NCF can be uniquely written as 2.1 and the number r is uniquely determined by f, we have

Definition 2.2. *For a NCF f written as equation 2.1, the number r will be called its LAYER NUMBER. Variables of M_1 will be called the most dominant variables(canalyzing variable), they belong to the first layer of this NCF. Variables of M_2 will be called the second most dominant variables and belong to the second layer of this NCF and etc. We Call $[k_1, \ldots, k_r]$ the profile of f. There are 2^{n+1} NCFs with the same profile.*

3. Sensitivity and block sensitivity of NCF

Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}^n$, $[n] = \{1, \ldots, n\}$. For any subset S of $[n]$, we form $\mathbf{x}^S$ by complementing those bits in $\mathbf{x}$ indexed by elements of S. We write $\mathbf{x}^i$ for $\mathbf{x}^{\{i\}}$.

Definition 3.1. *The sensitivity of f at $\mathbf{x}$, $s(f; \mathbf{x})$, is the number of indices i such that $f(\mathbf{x}) \neq f(\mathbf{x}^i)$. The sensitivity of f, denoted $s(f)$, is $Max_{\mathbf{x}} s(f; \mathbf{x})$*

Definition 3.2. [17] *The block sensitivity of f at $\mathbf{x}$, $bs(f; \mathbf{x})$, is the maximum number of disjoint subsets $B_1, \ldots, B_r$ of $[n]$ such that, for all j, $f(\mathbf{x}) \neq f(\mathbf{x}^{B_j})$. We refer to such a set B_j as a block. The block sensitivity of f, denoted $bs(f)$, is $Max_{\mathbf{x}} bs(f; \mathbf{x})$.*

Definition 3.3. [11] *The l-block sensitivity of f at $\mathbf{x}$, $bs_l(f;\mathbf{x})$, is the maximum number of disjoint subsets $B_1, \ldots, B_r$ of $[n]$ such that, for all j, $B_j \leq l$ and $f(\mathbf{x}) \neq f(\mathbf{x}^{B_j})$. The l-block sensitivity of f, denoted $bs_l(f)$, is $Max_{\mathbf{x}} bs_l(f;\mathbf{x})$.*

Obviously, we have $0 \leq s(f) \leq bs_l(f) \leq bs(f) \leq n$.

Lemma 3.4. *Let σ be a permutation on $[n]$, and $(a_1 \ldots, a_n) \in \mathbb{F}^n$, Let $g = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ and $h = f(x_1 \oplus a_1, \ldots, x_n \oplus a_n)$. Then the sensitivity, l-block sensitivity and block sensitivity of f, $f \oplus 1$, g, h are same.*

Proof. This follows from the above definitions. □

Because of Lemma 3.4, In the rest of this section, for NCF in equation 2.1, we always assume

$$f(x_1, x_2, \ldots, x_n) = f_r = M_1(M_2(\ldots(M_{r-1}(M_r \oplus 1) \oplus 1)\ldots) \oplus 1)$$

and $M_1 = x_1 \ldots x_{k_1}$, $M_2 = x_{k_1+1} \ldots x_{k_1+k_2}$,…, $M_r = x_{k_1+\ldots+k_{r-1}+1} \cdots x_n$
Let $\mathbf{x} = (\mathbf{x_1}, \ldots, \mathbf{x_r})$, where $\mathbf{x_i}$ has k_i bits, $i = 1, \ldots, r$. We have

Lemma 3.5. *For NCF f, for any word $\mathbf{x}$, if there are more than one zero bits in a subword $\mathbf{x_i}$, then we keep only one zero and flip the others to get a new word $\mathbf{x}'$, we have $s(f;\mathbf{x}) \leq s(f;\mathbf{x}')$(In fact, $s(f;\mathbf{x}') = s(f;\mathbf{x})$ or $s(f;\mathbf{x}') = s(f;\mathbf{x}) + 1$) and $bs(f;\mathbf{x}) = bs(f;\mathbf{x}')$*

Proof. There are at least two zeros in $\mathbf{x_i}$, so M_i is always zero (hence, does not change) even if one of the bits is flipped. hence, $s(f;\mathbf{x}) \leq s(f;\mathbf{x}')$.

Let $bs(f;\mathbf{x}) = t$, and B_j, $j = 1, \ldots, t$ be the blocks such that $f(\mathbf{x}^{B_j}) \neq f(\mathbf{x})$. We can assume that each block B_j is minimal, i.e., for any proper subset $B'_j \subset B_j$, $f(\mathbf{x}^{B'_j}) = f(\mathbf{x})$. Suppose there is a block B_{j_0} involves the bits in $\mathbf{x_i}$, it means it changes the value of M_i from 0 to 1. It must change all the zero bits in $\mathbf{x_i}$ to 1. Such B_{j_0} is unique since all the blocks are disjoint. We can construct the block B'_{j_0}(a subset of B_{j_0}), which has only one index whose corresponding bit is the only zero bit of $\mathbf{x'_i}$. Take all the other blocks same as B_j ($j \neq j_0$). We get the value of $bs(f, \mathbf{x}') \geq t = bs(f, \mathbf{x})$. On the other hand, there are more zeros in $\mathbf{x_i}$, in order to change the value of M_i (hence, a possible change of f) from 0 to 1, it needs to change more than one bits, hence the number of maximal blocks will be probably less (or same), i.e., we have $bs(f, \mathbf{x}) \geq bs(f, \mathbf{x}')$. Hence, $bs(f, \mathbf{x}) = bs(f, \mathbf{x}')$ □

We are ready to prove the main result of this paper, we have

Theorem 3.6. *f_r is nested canalyzing with profile $[k_1, \ldots, k_r]$, then*

$$s(f_1) = n. \text{ For } r > 1,\ s(f_r) = \begin{cases} Max\{k_1 + k_3 + \ldots + k_r, k_2 + k_4 + \ldots k_{r-1} + 1\}, 2 \nmid r \\ Max\{k_1 + k_3 + \ldots + k_{r-1} + 1, k_2 + k_4 + \ldots k_r\}, 2 | r \end{cases}$$

Proof. It is obvious that $S(f_1) = n$.

For $r > 1$, we first consider that r is odd.

Let $s(f_r, \mathbf{x})$ be the sensitivity of f_r on word $\mathbf{x}$ for $\mathbf{x} = (\mathbf{x_1}, \ldots, \mathbf{x_r})$. Because of Lemma 3.5, in order to find the maximal value, we can assume that there is either no zero or exactly one zero bit in every $\mathbf{x_i}$.

In the following, we consider all the possibilities of such words $(\mathbf{x_1}, \ldots, \mathbf{x_r})$.

Case 1: One zero in $\mathbf{x_1}$.

$f = 0$, in order to change the value, the zero bit in $\mathbf{x_1}$ must be changed. Hence, $S(f_r, \mathbf{x}) \leq 1$.

Case 2: No zero in $\mathbf{x_1}$, but one zero in $\mathbf{x_2}$.

$f_r = M_1(M_2(\ldots) \oplus 1) = M_1 = 1$, the value of f_r does not change by flipping any bit in $\mathbf{x_i}$($i \geq 3$)or any nonzero bits in M_2. Hence, $s(f_r, \mathbf{x}) \leq k_1 + 1$.

Case 3: No zero in $\mathbf{x_1}$ and $\mathbf{x_2}$, but a zero in $\mathbf{x_3}$.

$f_r = M_1(M_2(M_3(\ldots) \oplus 1) \oplus 1) = M_1(M_2 \oplus 1) = 0$. In order to change the value of f_r (from 0 to 1), we can only flip the bits in $\mathbf{x_2}$ or possibly the zero bit in $\mathbf{x_3}$, hence, $s(f_r, \mathbf{x}) \leq k_2 + 1$.

Case 4: No zero in $\mathbf{x_1}$, $\mathbf{x_2}$ and $\mathbf{x_3}$, but a zero in $\mathbf{x_4}$.

$f_r = M_1(M_2(M_3 \oplus 1) \oplus 1) = 1$. We can change the value of f_r (from 1 to 0)by flipping any bit in $\mathbf{x_1}$ or $\mathbf{x_3}$(or possible the zero bit in $\mathbf{x_4}$) but not the bit in $\mathbf{x_2}$ and all the $\mathbf{x_i}$ with $i \geq 5$. Hence, we have $S(f_r, \mathbf{x}) \leq k_1 + k_3 + 1$.

$\ldots\ldots\ldots$

Case r: No zero in $\mathbf{x_i}$, $i = 1, \ldots, r-1$ but one zero in $\mathbf{x_r}$.

$f_r = M_1(M_2(\ldots(M_{r-1} \oplus 1)\ldots) \oplus 1) = 0$ (since $r-1$ is even). We can change the value of f_r (from 0 to 1) by flipping one bit of any $\mathbf{x_2}$, $\mathbf{x_4}$, …, $\mathbf{x_{r-1}}$ and the zero bit of $\mathbf{x_r}$ but not the bit of $\mathbf{x_1}$, $\mathbf{x_3}$, …, $\mathbf{x_r}$. Hence, we have $S(f_r, \mathbf{x}) = k_2 + k_4 + \ldots + k_{r-1} + 1$

Case $r+1$: No zero in $\mathbf{x_i}$, $i = 1, \ldots, r$, i.e., $\mathbf{x} = (1, \ldots, 1)$. $f_r = M_1(M_2(\ldots(M_r \oplus 1)\ldots) \oplus 1) = 1$ (since r is odd). We can change the value of f_r (from 1 to 0) by flipping one bit of any $\mathbf{x_1}$, $\mathbf{x_3}$, …, $\mathbf{x_r}$ but not the bit of $\mathbf{x_2}$, $\mathbf{x_4}$, …, $\mathbf{x_{r-1}}$. Hence, we have $S(f_r, \mathbf{x}) = k_1 + k_3 + \ldots + k_r$

In summary, we have $s(f_r) = Max\{k_2 + k_4 + \ldots + k_{r-1} + 1, k_1 + k_3 + \ldots + k_r\}$.

When r is even, the proof is similar.

□

Corollary 3.7. $s(f_1) = n$. $s(f_{n-1}) = \begin{cases} \frac{n+2}{2}, 2|n \\ \frac{n+1}{2}, 2 \nmid n \end{cases}$.

If $2 \leq r \leq n-2$, $\frac{n+1}{2} \leq S(f_r) \leq \begin{cases} n+1-\frac{r+1}{2}, 2 \nmid r \\ n+1-\frac{r}{2}, 2|r \end{cases}$

Proof. Because $(k_2 + k_4 + \ldots + k_{r-1} + 1) + (k_1 + k_3 + \ldots + k_r) = n + 1$, hence, $Max\{k_2 + k_4 + \ldots + k_{r-1} + 1, k_1 + k_3 + \ldots + k_r\}| \geq \frac{n+1}{2}$. By considering the two minimal possibilities of $(k_2 + k_4 + \ldots + k_{r-1} + 1)$ and $(k_1 + k_3 + \ldots + k_r)$, we will get the maximal valus of these two numbers. Hence, we can get the other side of the above inequality.

□

In the following , we will prove the block sensitivity of any NCF is same to its sensitivity. Because of Lemma 3.4, we still assume

$$f(x_1, x_2, \ldots, x_n) = f(\mathbf{x}) = f(\mathbf{x_1}, \ldots, \mathbf{x_r}) = f_r = M_1(M_2(\ldots(M_{r-1}(M_r \oplus 1) \oplus 1)\ldots) \oplus 1)$$

and $M_1 = x_1 \ldots x_{k_1}$, $M_2 = x_{k_1+1} \ldots x_{k_1+k_2}, \ldots$, $M_r = x_{k_1+\ldots+k_{r-1}+1} \ldots x_n$

Theorem 3.8. *Let f be any NCF, then $s(f) = bs(f)$.*

Proof. Actually, by Lemma 3.5, we just need to prove $s(f_r; \mathbf{x}) = bs(f_r; \mathbf{x})$ for any $\mathbf{x}$ such that there is at most one zero bit in each subword $\mathbf{x_i}$. If $r = 1$, since $s(f_1) = n \leq bs(f_1) \leq n$, we have $bs(f_1) = n$. In the following we assume $r \geq 2$. For any word $\mathbf{x}$, let the first zero bit of $\mathbf{x}$ appear in $\mathbf{x_i}$, i.e., $M_1 = \ldots = M_{i-1} = 1$ and $M_i = 0$. So, we have

$$f_r = M_1(M_2(\ldots(M_{i-2}(M_{i-1} \oplus 1) \oplus 1)\ldots) \oplus 1) = \begin{cases} 1, 2|i \\ 0, 2 \nmid i \end{cases} \tag{3.1}$$

Let $bs(f_r; \mathbf{x}) = t$, and B_j, $j = 1, \ldots, t$ be the disjoint blocks such that $f_r(\mathbf{x}^{B_j}) \neq f_r(\mathbf{x})$. We can assume that each block B_j is minimal, i.e., for any proper subset $B'_j \subset B_j$, $f_r(\mathbf{x}^{B'_j}) = f_r(\mathbf{x})$. First, all the blocks do not involve the bits of M_j with $j \geq i+1$ because of 3.1. To change the value of f_r, some M_l must be changed (from 1 to 0) for $l = 1, \ldots, i-1$ or M_i be changed from 0 to 1. In order to do so, we need only to flip one bit in M_l ($l = 1, \ldots, i-1$) from 1 to 0 or change the zero bit in M_i to 1. Hence the corresponding block B_j has only one index since it is minimal. We actually have proved $s(f_r; \mathbf{x}) \geq t$, hence $s(f_r; \mathbf{x}) = bs(f_r; \mathbf{x})$.

□

4. Monotone nested canalyzing functions

In this section, we determine all the functions which are both monotone and nested canalyzing.

Definition 4.1. *Let* $\mathbf{x} = (x_1, \dots x_n) \in \mathbb{F}_2^n$ *and* $\mathbf{y} = (y_1, \dots y_n) \in \mathbb{F}_2^n$*, we define* $\mathbf{x} \prec \mathbf{y}$ *iff* $x_i \leq y_i$ *for all* $i \in [n]$.

Definition 4.2. $f(\mathbf{x})$ *is monotone increasing (decreasing) if* $f(\mathbf{x}) \leq f(\mathbf{y})$ $(f(\mathbf{x}) \geq f(\mathbf{y}))$ *whenever* $\mathbf{x} \prec \mathbf{y}$.

Lemma 4.3. *If* f *is monotone increasing (decreasing), then fix the values of some bits, the remain function of the remaining variable is still monotone increasing (decreasing).*

Lemma 4.4. $f(\mathbf{x}) = (x_1 \oplus a_1) \dots (x_n \oplus a_n) \oplus b$ *is monotone iff* $a_1 = \dots = a_n$.

Lemma 4.5. f *and* g *are monotone increasing (decreasing) then* fg *is also increasing (decreasing).* $f \oplus 1$ *will be decreasing (increasing).*

Let f_r be a NCF and written as 2.1.

Theorem 4.6. f_r *is monotone iff* $M_i = \prod_{j=1}^{k_i}(x_{i_j} \oplus a)$ *and* $M_{i+1} = \prod_{j=1}^{k_{i+1}}(x_{i+1_j} \oplus \overline{a})$ *for* $i = 1, 3, 5, \dots$*. Where* $a \in \{0, 1\}$

Proof. By suitably fixing the values of the other variables, we can get $f_r = M_i \oplus 1$ for $i > 1$ or M_1. Hence, by Lemma 4.3 and Lemma 4.4, we have $M_i = \prod_{j=1}^{k_i}(x_{i_j} \oplus a)$. Again, we may suitably fix the values of the other variables to get $f_r = M_i(M_{i+1} \oplus 1)$. If $M_{i+1} = \prod_{j=1}^{k_i}(x_{i_j} \oplus a)$, $M_i(M_{i+1} \oplus 1)$ is not monotone. Hence, $M_{i+1} = \prod_{j=1}^{k_{i+1}}(x_{i+1_j} \oplus \overline{a})$.

On the other hand, use induction principle , it is easy to prove these NCFs are monotone with the help of the above three lemmas.

Actually, When $M_1 = x_1 \dots x_{k_1}$, f_r is increasing, when $M_1 = (x_1 \oplus 1) \dots (x_{k_1} \oplus 1)$, f_r is decreasing. □

Corollary 4.7. *The number of monotone nested canalyzing functions (MNCFs) is*

$$= 4 \sum_{\substack{k_1+\dots+k_r=n \\ k_i \geq 1, i=1,\dots,r-1, k_r \geq 2}} \frac{n!}{k_1! k_2! \dots k_r!} = 4 \sum_{\substack{k_1+\dots+k_r=n \\ k_i \geq 1, i=1,\dots,r-1, k_r \geq 2}} \binom{n}{k_1, \dots, k_{r-1}}.$$

Proof. From Equation 2.1, for each choice $k_1, \dots, k_r$, with condition $k_1 + \dots + k_r = n$, $k_i \geq 1$, $i = 1, \dots, r-1$ and $k_r \geq 2$,

there are $\binom{n}{k_1}$ many ways to form M_1,

there are $\binom{n-k_1}{k_2}$ many ways to form M_2,

$\dots$,

there are $\binom{n-k_1-\dots-k_{r-1}}{k_r}$ many ways to form M_r,

a has two choices

b has two choices.

Hence, the number is

$$4 \sum_{\substack{k_1+\dots+k_r=n \\ k_i \geq 1, i=1,\dots,r-1, k_r \geq 2}} \binom{n}{k_1} \binom{n-k_1}{k_2} \dots \binom{n-k_1-\dots-k_{r-1}}{k_r}$$

$$= 4 \sum_{\substack{k_1+\dots+k_r=n \\ k_i \geq 1, i=1,\dots,r-1, k_r \geq 2}} \frac{n!}{(k_1)!(n-k_1)!} \frac{(n-k_1)!}{(k_2)!(n-k_1-k_2)!} \dots \frac{(n-k_1-\dots-k_{r-1})!}{k_r!(n-k_1-\dots-k_r)!}$$

$$= 4 \sum_{\substack{k_1+\ldots+k_r=n \\ k_i\geq 1, i=1,\ldots,r-1, k_r\geq 2}} \frac{n!}{k_1!k_2!\ldots k_r!} = 4 \sum_{\substack{k_1+\ldots+k_r=n \\ k_i\geq 1, i=1,\ldots,r-1, k_r\geq 2}} \binom{n}{k_1,\ldots,k_{r-1}}.$$

□

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